

1. THE IMAGINARY

§1.1. Mathematics and Truth

“What is truth?” asked a famous Roman governor. Indeed, what *is* truth and how are we to know it? When we were young we soon learnt to tell the difference between truth and lies. Indeed we learnt to tell lies almost as soon as we could talk. “It wasn’t me – Sarah did it!”



As we got older we learnt that things are not always what they seem. Optical illusions, and the sleight of hand of a magician, fascinated us.

WHAT IS TRUTH?

As adults we’ve learnt that truth can be relative. Things are not always black and white. Even lies can be all shades of grey from despicable black to the purest of white.

Of all the subjects that we learnt at school, mathematics is the one where truth is most clearly defined. “What I like about mathematics,” I’m often told, “is that things are either right or wrong – you really know where you stand.”

Well, it's true isn't it? In our history essays it wasn't so important what conclusions we reached, we were told, but rather how well we supported them.

History isn't just about names and dates and 'facts' but more about explanations of why things happened the way they did. And *your* explanation may be quite different to *mine* yet be considered equally good. Even the facts of history undergo change as scholars revisit contemporary sources and discover that what we've been taught all these years was not actually the case.

Science is a very objective study, based as it is on observation and experiment. Yet how often has there been radical change there. The sun no longer travels round the earth as it did for centuries until Galileo. The atom is no longer an indivisible piece of matter. Light, which once travelled in a straight line, now curves in a gravitational field.

But the theorems of Euclid are still as valid as they were over two thousand years ago. With mathematics you know where you stand. Things are either true or false and when we prove that something is true that's the end of the matter. Or is it?

§1.2. Do Imaginary Numbers Exist?

People often ask “does God exist?” It’s generally agreed that one cannot prove that there is a God. Some people talk about having faith and say that they believe in God. Atheists believe that he doesn’t. Agnostics say that you can’t tell, and leave it at that.

What has this to do with mathematics? One of the themes of this book is to reflect on the similarities between mathematics and religion. In terms of truth you might think that they are poles apart. In mathematics everything that is believed to be true can be proved while religious truth is purely a matter of belief. But things are not quite as simple as they sound.

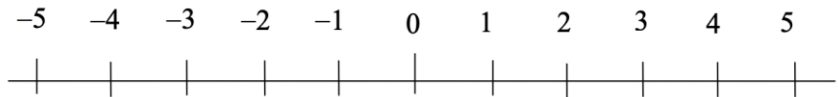
Now this is a book about mathematics not religion, so that if it should change your ideas about religious belief that’s your own business. My main purpose here is to show that there are many mathematical truths that seem to be contrary to intuition and that even the concept of mathematical truth itself is not quite what you might have imagined it to be.

The German mathematician Leopold Kronecker (1823–1891) once said “God created the natural numbers; all else is the work of man.” He meant that the so-called “natural numbers”, the ones that we count with, exist in an obvious way in the world around us. But fractions,

negative numbers and decimals are artificial constructions.

Mathematicians began to become bothered with existence of certain numbers in the 17th century when they made up some new numbers. Up to then it was believed that you can't have square roots of negative numbers.

At this stage numbers were synonymous with points on the number line. In the 'middle' of an infinitely long line you have the number zero. To the left are the negative numbers and to the right are the positive ones.



Whole numbers step out in both directions in uniform steps and fractions, and more generally decimals, fill in the space between them. To use the correct mathematical terms we have the **integers** represented by points made by equal sized steps and **rational numbers**, and more generally **real numbers**, fill up the number line.

There are certain facts that can be proved about these numbers. One of these is that if you multiply two negative numbers you get a positive one. And of course if you multiply two positive numbers you get a positive one.

It follows that negative numbers don't have square roots. If you multiply a number by itself the answer will

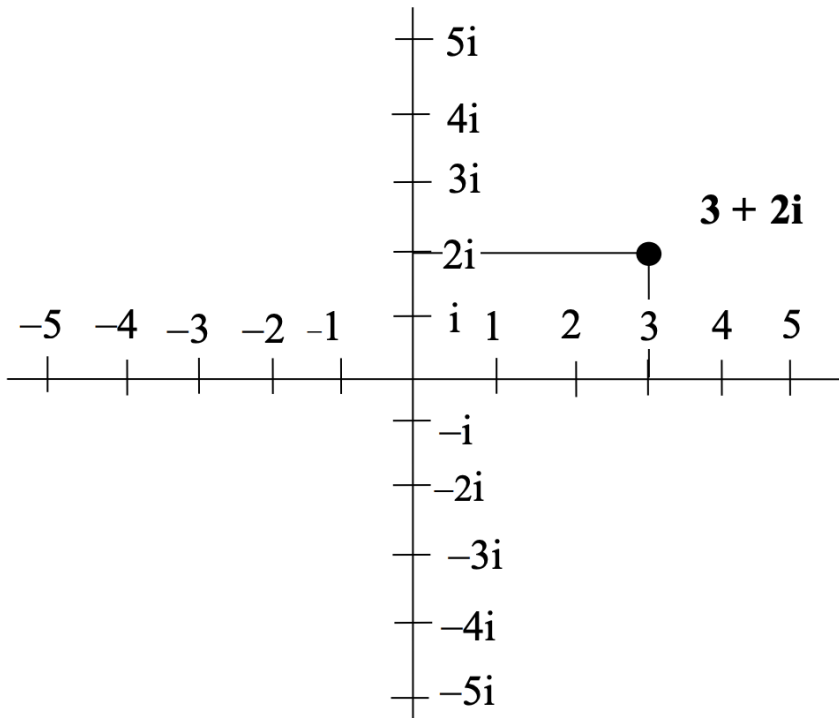
always be positive (or zero if you square 0). Here's something which was once considered to be a true fact – but not any more!

In the 17th century, in order to solve certain practical problems, mathematicians found it useful to *invent* square roots for negative numbers. These were called **imaginary numbers** because it was felt, useful as they are, that they didn't really exist. The square roots of -1 were called i and $-i$, with the symbol 'i' being used to remind us that they are only 'imaginary' – they don't really exist.

But how did they reconcile this with the fact that if this imaginary number i is either positive or negative its square must be positive. How could it possibly be equal to -1 ? Well, who's to say that this imaginary number i must be either positive or negative? Can't it be neither?

Today we live in a complex world where the simple axiom "everybody is either male or female" no longer holds. Why can't non-zero numbers be neither positive or negative? But surely the point that represents a non-zero number must be either to the left or the right of zero. On a line there are only two directions, left or right. That's true, but in a plane you can also go up and down. To accommodate these imaginary numbers we need to go into two dimensions.

If we're going to invent the imaginary number i we must allow it to combine arithmetically with the ordinary real numbers, so that we must invent numbers such as $2i$ and $3 + 2i$. More generally we'll have invented numbers of the form $a + bi$ where a and b are ordinary real numbers. These are called **complex numbers**. The name 'complex' doesn't refer to the level of difficulty but simply to the fact that these numbers have a complex structure, being made up of two parts. This diagram shows how every complex number can be placed on what is called the **complex plane**.



It might seem a cheat when asked to add the numbers 3 and $2i$ and to be told that it is $3 + 2i$. Is this the question or is it the answer? In fact it's both. It's not possible to simplify this answer. It is just like asking someone to divide 2 by 3 and being told that the answer is $\frac{2}{3}$.

Doing arithmetic with these complex numbers is not difficult. To add $3 + 2i$ to $5 + 7i$ we get $8 + 9i$. And $(2 + 3i)(5 + 7i) = 10 + 14i + 15i + 21i^2$. Remembering that $i^2 = -1$ we can simplify this to $10 + 29i - 21 = -11 + 29i$. Division is a little bit trickier.

These complex numbers proved to be extremely useful. Indeed modern electronics couldn't exist without them. But the philosophic question remains "do these imaginary numbers really *exist*?" Mathematicians used to struggle over such questions but the modern mantra is "if it doesn't exist then you just invent it".

So when faced with fact that parallel lines in a plane don't meet, mathematicians just invented new points where they *do* meet and so created a new geometry called the **Projective Plane**. There's a sense in which mathematics is just a game of make-believe! But it has proved to be such a useful make-believe that we don't fuss about whether these made up entities exist. It's a non-question.

Many people say that God is just something made up by people in order to explain the natural phenomena such as thunder (that was before science) or to make sense of their lives. Others say that God existed long before man was capable of thinking about things. I leave it to you to make your mind up on that matter. I just want to point out that many mathematicians when they make up some mathematical entity that proves to be useful they get an uncanny feeling that they're not inventing but rather discovering something that was already there. In some sense complex numbers existed before the world began. If there's an advanced civilization on some far distant planet they will also have 'invented' complex numbers in one form or another.

§1.3. Mathematics Contains No Facts

Mathematics is the subject *par excellence* when it comes to logical foundations. Yet in another sense mathematics isn't about truth at all – certainly not in an absolute sense. When we prove that the angles of a triangle add up to 180° we think we've proved an absolute truth about the real world. Not so!

Mathematics is not about absolute truth (if there is such a thing) but rather about relative truth. Everything in mathematics is based on definitions and fundamental assumptions. Take the case of the angle sum theorem we've just mentioned. Quite apart from needing to define

angles and triangles and addition we must accept the axioms on which the proof of the theorem is based.

Euclid began by setting down some basic axioms, or assumptions. Some of these were attempts at definitions, others were considered as “self-evident truths”. Clearly, through any two distinct points there’s exactly one straight line. You don’t need to prove it – you can see that it’s obvious. If anyone is so obtuse as to say they don’t agree with it you simply have to ask them to put two points on a piece of paper and draw two *different* straight lines between them.

But our grounds for accepting this axiom are rather shaky. We’re arguing as a physicist might. We carry out many experiments with points and lines on a sheet of paper and are never able to construct two straight lines between the same two points. I’m not belittling the scientific method, but if we allow it to operate within geometry we may just as well go off and measure lots of triangles and conclude the truth of the angle sum theorem by experiment.

We might argue that light travels in straight lines so a ray of light that begins at point A and is seen at point B must have gone along a single path. Imagine if the light had to make up its mind as to which straight line to follow!



Unfortunately we now know that light doesn't always travel exactly in straight lines. The more gravitation is around the more curved the path. And as for light not being able to make up its mind as to which path to follow, even stranger things have been observed in the laboratory since the advent of quantum physics.

Another fundamental 'truth' is that if we have a straight line, and a point off that line, there's exactly one line passing through the point parallel to the given line. It is on the basis of this that the Angle Sum Theorem is proved.



Now experimental evidence for this 'fact' is very strong. But remember that drawing lines on a piece of paper is neither particularly accurate nor particularly general. Perhaps two lines can be drawn, so close to each other that you'd only notice the difference if they were drawn with considerable precision and the sheet of paper was many light years across. Indeed there's speculation

that the geometry of space doesn't quite follow Euclid's axioms.

How do mathematicians cope with all this? Scrap thousands of years of Euclidean geometry? Not at all! "There's nothing wrong with Euclidean geometry," they say. "If the axioms hold then so do the theorems. It's the job of the cosmologist to decide whether the axioms are true for our universe, not the mathematician!"

What mathematicians did do, when it was discovered that this 'fact' didn't follow logically from the other axioms, was to develop non-Euclidean geometries where there can be more than one line through two distinct points, or none at all. So by the time physicists began to doubt whether our universe followed Euclidean geometry there was a mature study of non-Euclidean geometry for them to choose as an alternative.

§1.4. The Disembodied Angel

Mathematics isn't about absolute truth.
Mathematicians create stories about imaginary systems. Each one is logically consistent but it's up to the physicist, or economist, to select one off the shelf to fit their observations.

If the universe were to disappear tonight, physics and chemistry would be no more. Biology and psychology would disappear, not to mention economics.

Of all branches of learning only mathematics (and perhaps theology) would remain! It's a nice thought, though whether logic exists outside the hard-wiring of the human brain is yet another question. But certainly a mathematical truth shouldn't be dependent on physical observation. After all we must think of the disembodied angels!

Years ago one of my colleagues, Alan Macintosh, had to teach an advanced class in geometry. To emphasise the fact that geometry can be studied without recourse to spatial intuition he had a pair of walkie-talkies (these day's he'd use mobile phones). An accomplice was positioned in the next classroom with one handset and Professor Macintosh had the other. A student from the class was chosen and was given the job of explaining some geometrical concept to the 'disembodied angel' in the other room. The idea was that the 'angel' was infinitely intelligent but had no concept of space, living as she did in a purely spiritual realm. The results were quite amusing.

Mathematics has reached the level of maturity that it can now be taught to disembodied angels! That is, when studying it at an advanced level, students are required to empty themselves of all their intuition concerning number, space and even sets. The fundamental objects of study are to be considered as undefined entities. We have to capture our intuition by writing down our assumptions

as axioms. They might be self-evident to us but not to disembodied angels. Both they and we accept these axioms and proceed from there. At no time in the proofs of our theorems must we fall back on intuition. Everything must proceed using the tools of logic.

That's not to say that intuition has no place in modern mathematics. Mathematicians are not machines that manipulate symbols mindlessly to create theorems. There's an old joke that mathematicians are machines for turning coffee into theorems but this perhaps reflects the fact that coffee may help to stimulate a mathematician's intuition. What happens is an interaction between intuition and logic. An intuitive insight causes a mathematician to 'see' that such and such must be true. He, or she (women are now quite active in the world of research mathematics), will then set out to prove the fact, using sound logic. Sometimes they will fail, but their efforts will help them to see that they were wrong. More often than not they will be able to prove that they were right. Either way the struggle towards a proof will strengthen their intuition.

For some laymen, the phrase 'mathematical research' is an oxymoron. I am often asked "Hasn't everything in mathematics been discovered a long time ago?" Well, mathematics may not be quite the oldest profession but it comes close. It's probably the oldest academic profession.

Mathematics has been building for thousands of years. And because it's highly structured you can only understand the more recent bits once you understand the earlier bits. So all of the mathematics you learnt at school, even at the most advanced level, would be a couple of hundred years old. If you continued on to university mathematics you might be brought up to the end of the nineteenth century, with a couple of exceptions, though you'd still learn only a tiny fraction of what was known to that point.

But mathematics has been producing new theorems, even whole new branches, at an ever increasing rate. Some years ago, before reviews of mathematical papers went online, *Mathematics Reviews* was putting out monthly volumes, each the size of a small telephone book, that contained short summaries of the more important mathematics papers that had been published that month in research journals around the world.

§1.5. Propositional Logic

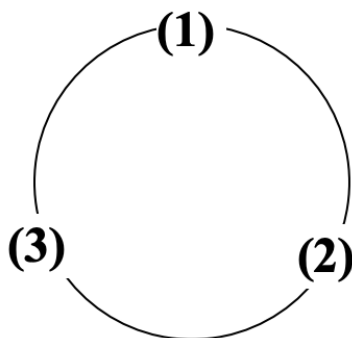
So mathematics is founded on logic and uses its tools to create proofs. What are the tools of logic? To begin with we consider things called 'statements' or 'propositions'. We could regard a statement as an undefined entity but it helps to think of it as a sentence for which it is meaningful to say that it is true or that it is false. Statements have things called 'truth values' – TRUE and FALSE.

Not every sentence is TRUE or FALSE. “Come here!” is a command, not a statement. But even things that look like statements might just be pretending. Consider the sentence: “THIS STATEMENT IS FALSE”.

If it’s TRUE then it’s FALSE and if it’s FALSE it must be TRUE. So it can be neither TRUE nor FALSE. This doesn’t invalidate logic – it simply means that “THIS STATEMENT IS FALSE” isn’t really a statement.

The reason why “THIS STATEMENT IS FALSE” can’t be considered a statement is because it is *self-referential* – that is it refers to itself. A similar situation exists if we have a collection of statements that refer to other statements, but where the references go around in a circle such as:

- (1) Statement (2) is FALSE**
- (2) Statement (3) is FALSE**
- (3) Statement (1) is FALSE**



If any of these is TRUE, the next, around the circle is FALSE.

If any of these is FALSE, the next, around the circle is TRUE.

So the TRUE and FALSE tags must alternate, which around a circle with an odd number of points is impossible.

You might think that the recipe to avoid such paradoxes is to insist that no statement is allowed to refer to itself either directly or indirectly. But there are instances where there's no hint of self-referentiality and where we still get such a paradox. We'll see such a system in a later chapter. This shows that it's very difficult to define what is meant by a statement.

We represent statements by lower case letters p , q , ... It's just like algebra except that the symbols stand for statements instead of numbers. The only property of a statement that logic can deal with is its truth value. Whether the statement is long-winded or amusing, or contains a certain four letter word, lies outside the realm of logic. So you can think of the variables p , q , r , ... as undefined entities having one of two possible values T or F (shorthand for TRUE and FALSE).

Logic thrives on constructing complex statements from simple ones, and then asking whether the complex statement is true or false. It does this using 'logical operators'.

The basic one is **‘not’**. When we say ‘not p ’ we mean ‘ p is FALSE’. So if p is TRUE then not p is FALSE but if p is FALSE then not p is TRUE.

For example if $p = ‘2 + 2 = 4’$ (TRUE) then not $p = ‘2 + 2 \neq 4’$ (FALSE) while if $q = ‘3 > 9’$ (FALSE) then not $q = ‘3 \leq 9’$ (TRUE).

If $g = “\text{God exists}”$ then not $g = “\text{God does not exist}”$. In this case you must decide for yourself whether p or not p is TRUE. We can’t have both being TRUE and our logic insists that at least one of them is TRUE. (There are other logics that logicians study where statements may be neither, but mathematicians are usually intuitive about our logic.)

Two statements p and q can be combined in several ways: **‘and’**, **‘or’** or **‘implies’**.

The complex statement p and q means what it says – we assert that both of them are TRUE. We can express this to a disembodied angel by means of a table that sets out the truth value of p and q under all four combinations of the truth values of p and q separately. Such an angel doesn’t need to have any concept of what ‘TRUE’ means.

p and q		
\downarrow p q \rightarrow	T	F
T	T	F
F	F	F

Sometimes we say p but not q . We might say “mathematics is interesting but economics is not”. Here ‘but’ just means ‘and’, at least at the basic level of logic. There may be overtones of surprise or contrast but such subtlety is beyond basic logic.

So if p is “mathematics is interesting” and q = “economics is interesting” then what we’re saying can be encapsulated in symbols as p and not q .

Or we might say p or q . Here our intuitive grasp of the word ‘or’ can more or less define what we mean. But there’s some ambiguity. There is the *exclusive* ‘or’ and the *inclusive* ‘or’.

At a party, if we’re offered a glass of wine, and are asked whether we want red or white, our host would be quite taken aback if we said “both”. Here the word ‘or’ is used in a polite sense, that is, it means the *exclusive* ‘or’. But mathematicians are impolite. We reserve the right to say “both” – perhaps not in a social situation but in our mathematics. When we say “ $x = 0$ or $y = 0$ ” we include three possibilities:

- (1) x is zero but y is not,
- (2) y is zero but x is not,
- (3) they are both zero.

Of course there are situations when we need to be exclusive, but then we'd have to spell it out:

$$p \text{ or } q \text{ and not}(p \text{ and } q).$$

For the benefit of the disembodied angel we should simply set out our meaning in a table.

p or q		
$\downarrow p \quad q \rightarrow$	T	F
T	T	T
F	T	F

The third logical operator is rather more confusing: “if p then q ”. We call this ‘implication’ but we don’t mean to imply any causal connection between the two – simply a connection between their truth values.

Let’s see how far our intuition might go to defining implication. In the case where p is TRUE and q is TRUE then of course we want “if p then q ” to be TRUE. True statements imply true statements.

And in the case where p is TRUE and q is FALSE we want “if p then q ” to be FALSE. True statements don’t imply false ones.

What do FALSE statements imply? We may be tempted to say “nothing”. In other words we may think we want “if p then q ” to be FALSE whenever p is FALSE. But do we? Look at the table that would result from that decision.

$\downarrow p \quad q \rightarrow$	T	F
T	T	F
F	F	F

Our disembodied angel would say, “This is the same table that you gave me for ‘and’. Do you mean that ‘if p then q ’ is just a complicated way of saying ‘ p and q ’?”

Rather than try to tease anything more out of our intuition we’ll simply present the correct table and be done with it. As Humpty Dumpty said in *Alice’s Adventures in Wonderland*, “when I use a word it means just what I choose it to mean.” Just accept that in mathematics “if ... then” means this.

if p then q		
$\downarrow p \quad q \rightarrow$	T	F
T	T	F
F	T	T

But surely it's wrong, surely, to have a FALSE statement implying anything! There's a technical explanation that doesn't become apparent until we meet quantifiers. At this stage just pretend that you're a disembodied angel and simply accept the table.

§1.6. Quantifier Logic

Here we move up to the next level of logic. It's going to involve some symbols so perhaps you're ready to skip to the end of the chapter. By all means, if you've got symbol phobia, then do just that. But let me encourage you to persist. Just remember that symbols are just short words and have to be read more slowly than most text.

In a short while we'll encounter the sentence

$$x^2 - y^2 = (x + y)(x - y).$$

Perhaps you might call it an "equation" but equations are really sentences. They have verbs and nouns. The verb in an equation is the symbol "=" that is shorthand for "equals". The nouns in the above sentence are x and y .

Now we could avoid using symbols and write this equation as a sentence in ordinary English. It says that "if you take any two numbers, square each of them and subtract the second number squared from the first number squared you'll always get the same answer as if you added the original two numbers, then calculated the first number minus the second and finally multiplied the sum by the

difference.” Do you really think that this makes it any easier to understand? Symbols are used in mathematics not to scare away the uninitiated but to make life easier.

What might scare you at first are the two strange symbols that are used to represent quantifiers. But quantifiers are things you use in everyday speech. It’s just that you probably don’t know the technical jargon for them, or the symbols that represent them.

Without quantifiers there’d be no mathematics. Come to think of it, without quantifiers our everyday conversation would be at the level of a caveman’s grunt.

“Children of today don’t know what hardship is”. Here we’re not referring to a particular child but to children in general. “Somebody strangled Joseph Yorke.” We may not know exactly “who done it” but we’re sure that Yorke died of strangulation. There you have the two types of quantifier. You use them all the time!

In mathematics we mostly make general statements involving variables. It wouldn’t be edifying to come across a theorem that said “ $3456 + 9876 = 9876 + 6543$ ”. It’s not a theorem we’d use very often! On the other hand there is a theorem that says:

$$x + y = y + x \text{ for all numbers } x \text{ and } y.$$

Now notice that here we have two variables x and y . What's meant by this is that we can substitute any number for x and any number for y and we get a true result.

$$5^2 - 4^2 = 25 - 16 = 9 \text{ and } (5 + 4)(5 - 4) = 9 \times 1 = 9.$$

$$9^2 - 7^2 = 81 - 49 = 32 \text{ and } (9 + 7)(9 - 7) = 16 \times 2 = 32.$$

$$10^2 - 1^2 = 99 \text{ and } (10 + 1)(10 - 1) = 11 \times 9 = 99.$$

$$3^2 - 5^2 = -16 \text{ and } (3 + 5)(3 - 5) = 8 \times (-2) = -16.$$

Such a theorem is much more useful than a specific one without any variables. It represents infinitely many individual true statements all at once.

To make it clear that we mean for x and y to represent any number we use the **universal quantifier**. We don't always write it in but it's implied when we write such an equation. What we mean is:

“for all x and y , $x + y = y + x$ ”.

There's a special symbol for the universal quantifier, an upside down A: \forall . So the above statement could be written as $\forall x \forall y [x + y = y + x]$.

If xPy is any statement involving two free variables x, y we can write $\forall x \forall y [xPy]$. Don't worry at this stage what we mean by a “free” variable. For example, if we were talking about the capital cities of Australia, xDy might represent the statement “it is possible to drive from

x to y ". In this case $\forall x \forall y [xDy]$ would be a symbolic version of the statement "it is possible to drive from any capital city to any capital city". You might have translated it as "... to any other" but, of course, you can drive from Sydney to Sydney, and so on".

The statement $\forall x \forall y [xPy]$ represents many individual statements, such as "it is possible to drive from Sydney to Perth". Now, if you are worried about driving from Sydney to Hobart, then don't. You won't have to. The fact that Hobart is in Tasmania, and Tasmania is an island separate from the mainland of Australia, simply means that $\forall x \forall y [xPy]$ is false.

The other quantifier is the **existential quantifier** written \exists . $\exists x [Px]$ means "for some x Px is TRUE". "Some" here means "at least one". So if Px means "x is reading this book" then $\exists x [Px]$ means that someone is reading this book. I guess that must be TRUE with you as a possible "x". But to say $\forall x [Px]$ would be to say that everyone is reading this book which I'm sure is FALSE.

Now in this example the x represents a person while in the more mathematical examples the x and y represent numbers. Behind the use of quantifiers is an understood **universe of quantification**.

If Mx meant "x was born of a mother" then it should be true that $\forall x Mx$, provided we understand the universe of quantification to be "all people". But it is false if $x =$

2. The number 2 was not born of a mother, and it is false if $x = \text{God}$, assuming that God exists of course!

Mathematical examples of the universe of quantification would be the set of all real numbers, the set of all complex numbers or the set of all triangles in the Euclidean plane. But we get some more human examples if we take the universe to be the set of all “beings”. Notice that we’ve gone beyond the set of all people so that we can include a possible God, and perhaps all the disembodied angels. We should perhaps define a “being” as an entity capable of loving.

Suppose that L represents “loves” so that xLy means “ x loves y ”. If $x = \text{your mother}$ and $y = \text{you}$, then hopefully xLy is TRUE. Consider what statements you can get by putting quantifiers in front of xLy .

To say $\forall x\forall y[xLy]$ would be to make the rather optimistic claim that everybody loves everybody. At the other extreme is the cynical claim that $\exists x\exists y[xLy]$. Somewhere, somebody loves somebody – the world is not totally devoid of love.

Of course if $\forall x\forall y[xLy]$ then $\exists x\exists y[xLy]$. If everybody loves everybody then certainly somebody loves somebody. But the latter statement is so much weaker that only a cynic would bother to say it.

It's a reasonable assertion that $\forall y \exists x [xLy]$ or "everybody is loved by somebody". Perhaps you can think of some exceptions but let's overlook the unlovable for the moment.

Notice what happens if we swap the quantifiers in the above statement. Understanding the difference between $\forall y \exists x [xLy]$ and $\exists x \forall y [xLy]$ gives mathematics students a great deal of trouble.

With our interpretation of xLy the statement $\exists x \forall y [xLy]$ means that "somebody loves everybody", that is there's a being who loves everybody. It has a sort of theological flavour to it. If there is a God, or at least a God of love, then $\exists x \forall y [xLy]$ would be true, and of course the weaker statement $\forall y \exists x [xLy]$ would also be true. But you may very possibly believe that everybody is loved by somebody but not believe that there is a single being who loves everybody.

A common logical construction in mathematics is the "if and only if". If we want to say that p is TRUE when q is TRUE and p is FALSE when q is FALSE we write " p if and only if q ". This is shorthand for making two statements "if p then q " and "if q then p ". As an example consider the following logical axiom.

not $\forall x [Px]$ if and only if $\exists x [\text{not } Px]$

In other words if it is FALSE that Px is not always TRUE then it must be sometimes FALSE. And conversely (this is the “if q then p ” bit) if something is sometimes FALSE then it isn’t always TRUE.

I’m sure you can see the sense in that, so why isn’t it a theorem? Using our intuition surrounding the meanings of the words it *is* obvious. But to the disembodied angel \forall and \exists are undefined things so the way they work have to be spelt out.

So we spell it out as an axiom of logic:
if $\exists x\forall y[xLy]$ then $\forall y\exists x[xLy]$

Here xLy would not just refer to the “ x loves y ” interpretation but to any statement involving two free variables.

Oh, what do we mean by a “free” variable? Simply one that isn’t “bound” by a quantifier. In the statement $\exists x[xLy]$ the variable x is bound but the y is free. It would be meaningful to put $y =$ Barrack Obama which would probably give a TRUE statement – “somebody loves Barrack Obama”.

But we could not put $x =$ Barrack Obama, or anyone else for that matter. When we write the statement $\exists x[xLy]$ as “someone loves y ” the “ x ” disappears altogether. It was only there as a “dummy variable”.

To make sure you've got the idea of quantifiers consider the following. We might define Hx to mean "x knows what hardship is" and write $\forall x[\neg Hx]$ to represent "children of today don't know what hardship is". Here our universe of quantification would be "children of today".

And if we used xSy to represent "x stabbed y" and let $y = \text{Yorke}$ then we might write $\exists x[xSy]$ to mean "somebody stabbed Yorke". Though why would we bother! In ordinary life we don't. Even mathematicians usually leave their quantifiers in the cupboard. But they *do* know their universals from their existentials!

It's amazing! You made it to the end of the chapter! It wasn't quite so readable as *Alice's Adventures in Wonderland* was it? The difference is that Lewis Carroll was content with throwing fragments of logic around in his delightful story. My aim is more ambitious. I want to take you on a real mathematical journey. The stories and poems between the chapters are just resting places.

And I hope you weren't put off by all the symbolic expressions. The secret is not to read mathematics as you'd read a novel. When you come to a symbolic expression you need to slow down and examine it symbol by symbol. It's a well-known fact that when reading English prose your eye can easily ignore a spelling error. You read whole words and if the word is misspelt you may

not even notice. (I bet you didn't even notice that "misspelt" was missing an "s".)

But you can't read an expression such as $\exists x\forall y[xLy]$ as a single word, otherwise you might miss the subtle difference between it and $\forall y\exists x[xLy]$. Anyway, your brain now needs a rest. That's the reason for the following Humpty Dumpty poem.

A POEM: HUMPTY DUMPTY



There existed an egg who sat on a wall,
And the wall being short implies this story is tall.
Now if that fat egg had had a great fall
Or slipped off the top, but not jumped, then not all
The king's horses and all the king's men,
If they worked through the day and the evening, then
They could not succeed if and only if when
They attempted to put Humpty together again.